Fractals

Section 10-18

Fractals

- “Regular” shapes vs. “irregular” shapes
- Pathological or monster curves
- Self-similarity

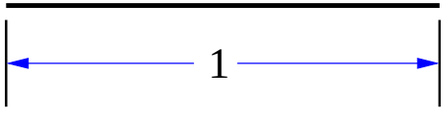
Euclidian Geometry

- Line
  - Unit: length (0 area, 0 volume)
  - 1D unit (ft, m, etc.)
  - Area of a line = 0, no matter how long it is
  - 2D measuring unit is not applicable

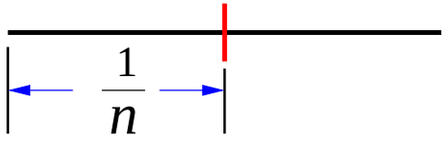
- Area
  - 2D unit (ft^2, m^2, etc.)
  - 1D unit of measure is not sufficient
  - An infinitely long line can be fitted within a small area

- Volume
  - 3D unit (ft^3, m^3, etc.)
  - 1D unit too “weak” to measure area
  - 3D unit too “strong” to measure area

1D in Euclidian Geometry



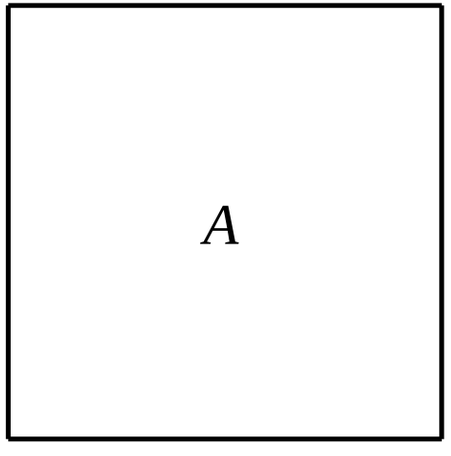
$D_E = 1$

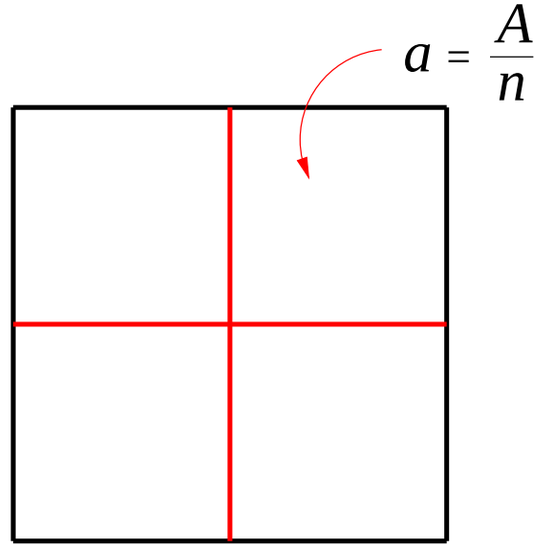


$s = \frac{1}{n}$ $n = 2$

$n s^1 = 1$

2D in Euclidian Geometry



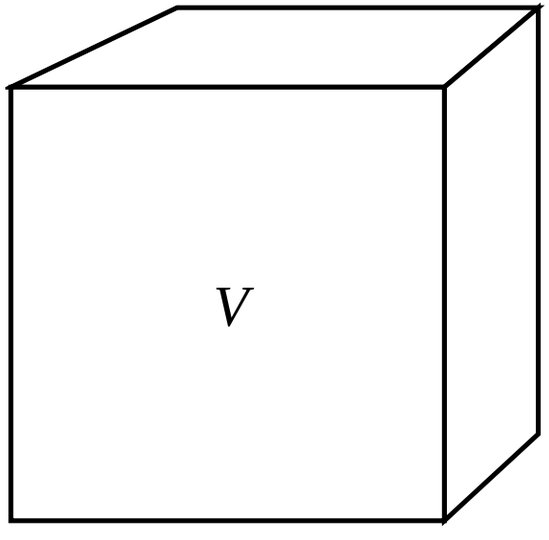


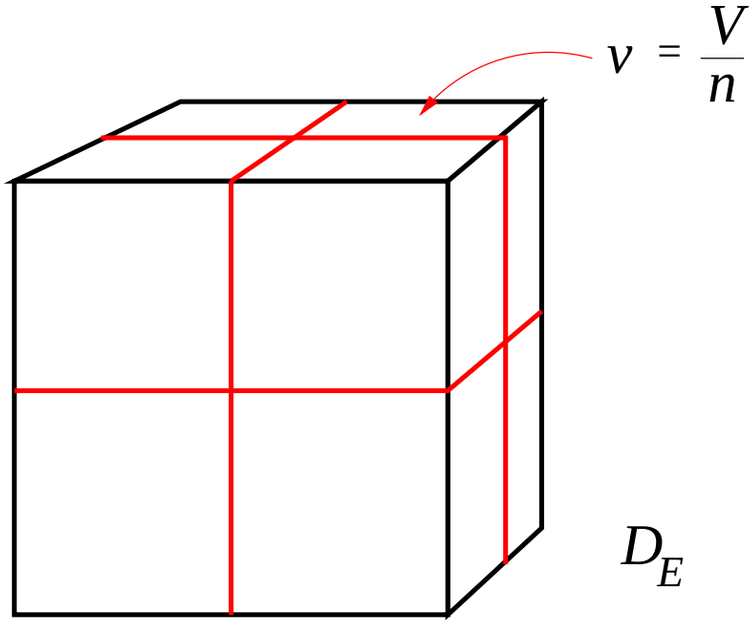
$D_E = 2$

$s = \frac{1}{n^{1/2}}$ $n = 4$

$n s^2 = 1$

3D in Euclidian Geometry

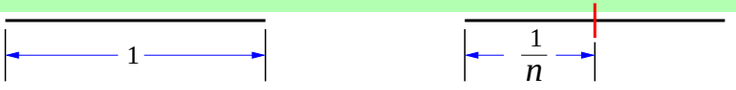




$D_E = 3$ $\quad s = \frac{1}{n^{1/3}}$ $\quad n = 8$

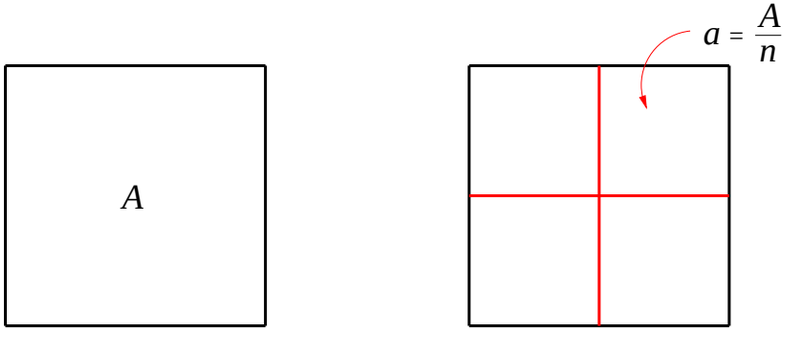
$n s^3 = 1$

Dimension in Euclidian Geometry



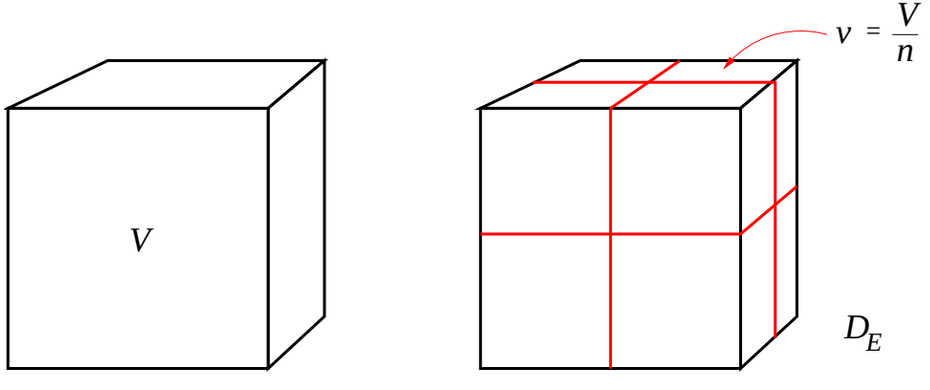
$D_E = 1$ $\quad s = \frac{1}{n}$ $\quad n = 2$

$n s^1 = 1$



$D_E = 2$ $\quad s = \frac{1}{n^{1/2}}$ $\quad n = 4$

$n s^2 = 1$



$D_E = 3$ $\quad s = \frac{1}{n^{1/3}}$ $\quad n = 8$

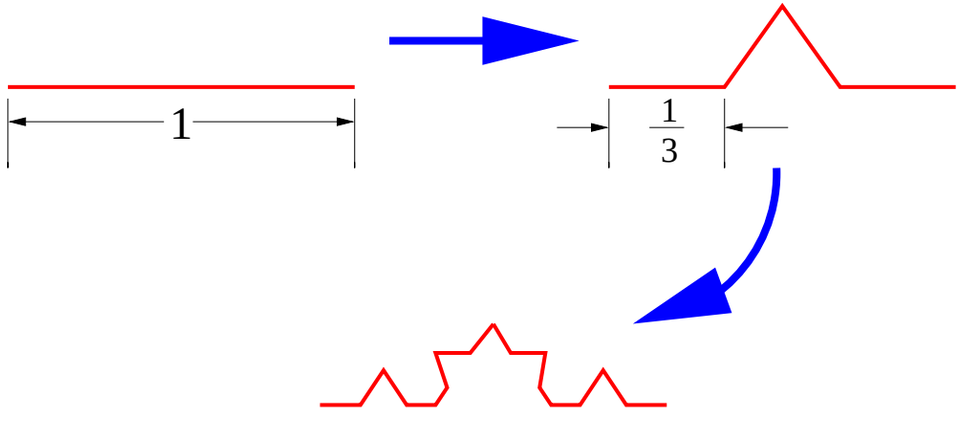
$n s^3 = 1$

Euclidean Dimension

- $ns^{D_E} = 1$
- Solve for D_E:

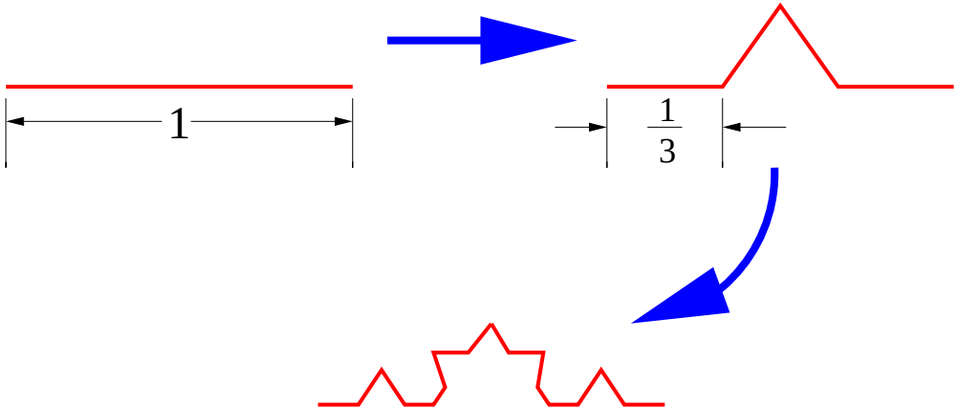
$$D_E = \frac{\log n}{\log \frac{1}{s}}$$

Koch Curve



Observe that the distance from one end point to another is finite but the length of the line itself is infinite!

Koch Curve



- $L(m) =$ length of curve with measuring unit m

$$\lim_{m \to 0} L(m) = \infty$$

- $L(\frac{1}{3}) = \frac{4}{3}$
- $L(\frac{1}{9}) = \frac{16}{9}$
- $L(\frac{1}{3^k}) = (\frac{4}{3})^k$

Fractal Dimension

- For the Koch curve:

$$ns^D = 1 \;\Rightarrow\; 4\left(\frac{1}{3}\right)^D = 1 \;\Rightarrow\; D = \frac{\log 4}{\log 3} = 1.2857$$

- The dimension is between 1 and 2
- Fractals have *fract*ional dimensions
- The Hausdorff-Besicovitch dimension $>$ the topological dimension
- $1 < D < 2 \;\Rightarrow$ fills more space than a line but less space than an area

Statistical Self Similarity

- Used to build natural looking rough surfaces